

# Explicit generators in rectangular affine $\mathcal{W}$ -algebras of type $A$

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## Abstract

We produce in an explicit form free generators of the affine  $\mathcal{W}$ -algebra of type  $A$  associated with a nilpotent matrix whose Jordan blocks are of the same size. This includes the principal nilpotent case and we thus recover the quantum Miura transformation of Fateev and Lukyanov.

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# 1 Introduction

Let  $\mathfrak{g}$  be a reductive Lie algebra over  $\mathbb{C}$  equipped with a symmetric invariant bilinear form and let  $f$  be a nilpotent element of  $\mathfrak{g}$ . The corresponding *affine  $\mathcal{W}$ -algebra*  $\mathcal{W}^k(\mathfrak{g}, f)$  at the level  $k \in \mathbb{C}$  is defined by the generalized quantized Drinfeld–Sokolov reduction; see [7], [10] and [11]. The case of a principal nilpotent element  $f$  was known much earlier and the corresponding  $\mathcal{W}$ -algebras were intensively studied; see [8, Ch. 15] for a detailed review of their structure, and [2] and [3] for representation theory.

In this paper we take  $\mathfrak{g} = \mathfrak{gl}_N$ . The Jordan type of a nilpotent element  $f \in \mathfrak{gl}_N$  is a partition of  $N$ . We will work with the elements  $f$  corresponding to partitions of the form  $(l^n)$  so that the associated Young diagram is the  $n \times l$  rectangle with  $nl = N$ . Our main result is an explicit construction of free generators of the  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}, f)$ . Moreover, we calculate the images of these generators with respect to the *Miura transformation*. In particular, if  $f$  is the principal nilpotent (i.e.,  $n = 1$ ) we thus reproduce the description of the  $\mathcal{W}$ -algebra due to Fateev and Lukyanov [6]. The results can be regarded as ‘affine analogues’ of the construction of the corresponding *finite  $\mathcal{W}$ -algebras* originated in [4], [13] and extended to arbitrary nilpotent elements  $f$  in [5].

## 2 Principal $\mathcal{W}$ -algebras

The case where  $f$  is a principal nilpotent will be useful in understanding arbitrary rectangular  $\mathcal{W}$ -algebras so we consider it first. We let  $e_{ij}$  denote the standard basis elements of  $\mathfrak{g} = \mathfrak{gl}_N$  and introduce subalgebras of  $\mathfrak{g}$  by

$$\mathfrak{b} = \text{span of } \{e_{ij} \mid i \geq j\}, \quad \mathfrak{m} = \text{span of } \{e_{ij} \mid i > j\} \quad \text{and} \quad \mathfrak{l} = \text{span of } \{e_{ii}\}.$$

Given  $k \in \mathbb{C}$  consider the normalized Killing form on  $\mathfrak{g}$  defined by

$$\kappa(x, y) = \frac{k}{2N} \text{tr}_{\mathfrak{g}}(\text{ad } x \text{ ad } y) = k \left( \text{tr}(xy) - \frac{1}{N} \text{tr } x \text{ tr } y \right), \quad x, y \in \mathfrak{g}. \quad (2.1)$$

Let  $\widehat{\mathfrak{b}} = \mathfrak{b}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$  be the Kac–Moody affinization of  $\mathfrak{b}$  with respect to the symmetric invariant bilinear form  $\kappa_{\mathfrak{b}}(x, y)$  on  $\mathfrak{b}$ , which is induced by (2.1) and given explicitly by

$$\kappa_{\mathfrak{b}}(e_{ii'}, e_{jj'}) = \delta_{ii'} \delta_{jj'} \left( k + N \right) \left( \delta_{ij} - \frac{1}{N} \right)$$

for  $i \geq i'$  and  $j \geq j'$ ; see [2, Lemma 4.8.1]. Furthermore, let  $V^k(\mathfrak{b})$  be the universal affine vertex algebra associated with  $\mathfrak{b}$  and  $\kappa_{\mathfrak{b}}$  [9]:

$$V^k(\mathfrak{b}) = U(\widehat{\mathfrak{b}}) \otimes_{U(\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C}, \quad (2.2)$$

where  $\mathbb{C}$  is regarded as the one-dimensional representation of  $\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1}$  on which  $\mathfrak{b}[t]$  acts trivially and  $\mathbf{1}$  acts as 1. By the Poincaré–Birkhoff–Witt theorem,  $V^k(\mathfrak{b})$  is isomorphic to

$U(\mathfrak{b}[t^{-1}]t^{-1})$  as a vector space. Choose the principal nilpotent element in the form

$$f = \sum_{i=1}^{N-1} e_{i+1\ i} \in \mathfrak{m}.$$

The principal  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}) = \mathcal{W}^k(\mathfrak{g}, f)$  can be realized as a vertex subalgebra of  $V^k(\mathfrak{b})$ ; see e.g. [2] and [8, Ch. 15]. Our aim is to give an explicit description of the generators of  $\mathcal{W}^k(\mathfrak{g})$  inside  $V^k(\mathfrak{b})$ .

Recall that the *column-determinant* of a matrix  $A = [a_{ij}]$  over an associative algebra is defined by

$$\text{cdet } A = \sum_{\sigma \in \mathfrak{S}_N} \text{sgn } \sigma \cdot a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(N)N}. \quad (2.3)$$

Introduce an extended Lie algebra  $\widehat{\mathfrak{b}} \oplus \mathbb{C}\tau$ , where the element  $\tau$  commutes with  $\mathbf{1}$ , and

$$[\tau, x[r]] = -rx[r-1] \quad \text{for } x \in \mathfrak{b} \text{ and } r \in \mathbb{Z}, \quad (2.4)$$

where  $x[r] = xt^r$ . In particular, (2.4) induces an associative algebra structure on the tensor product space  $U(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$ . Set  $\alpha = k + N - 1$  and consider the matrix

$$B = \begin{bmatrix} \alpha\tau + e_{11}[-1] & -1 & 0 & \cdots & 0 \\ e_{21}[-1] & \alpha\tau + e_{22}[-1] & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ e_{N-11}[-1] & e_{N-22}[-1] & \cdots & \alpha\tau + e_{N-1\ N-1}[-1] & -1 \\ e_{N1}[-1] & e_{N2}[-1] & \cdots & \cdots & \alpha\tau + e_{NN}[-1] \end{bmatrix}$$

with entries in  $U(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$ . For its column-determinant<sup>1</sup> we can write

$$\text{cdet } B = (\alpha\tau)^N + W^{(1)}(\alpha\tau)^{N-1} + \cdots + W^{(N)}$$

for certain coefficients  $W^{(r)}$  which are elements of  $U(\mathfrak{b}[t^{-1}]t^{-1})$ , and we can also regard them as elements of  $V^k(\mathfrak{b})$ . We can now state our main result on principal  $\mathcal{W}$ -algebras.

**Theorem 2.1.** *All coefficients  $W^{(1)}, \dots, W^{(N)}$  belong to  $\mathcal{W}^k(\mathfrak{g})$ . Moreover, they freely generate the  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}) \subset V^k(\mathfrak{b})$ .*

Before proving the theorem, note that the projection  $\mathfrak{b} \rightarrow \mathfrak{l}$  induces the vertex algebra homomorphism  $V^k(\mathfrak{b}) \rightarrow V^k(\mathfrak{l})$ , which restricts to the map

$$\nu : \mathcal{W}^k(\mathfrak{g}) \rightarrow V^k(\mathfrak{l}),$$

called the (*quantum*) *Miura transformation*. This is an injective vertex algebra homomorphism. The following formula for the images of the elements  $W^{(r)}$  under the Miura transformation is an immediate consequence of Theorem 2.1. It reproduces the construction of the  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g})$  due to Fateev and Lukyanov [6].

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<sup>1</sup>It is easy to verify that  $\text{cdet } B$  coincides with the *row-determinant* of  $B$  defined in a similar way.

**Corollary 2.2.** *Under the Miura transformation we have*

$$\sum_{r=0}^N \nu(W^{(r)})(\alpha\tau)^{N-r} = (\alpha\tau + e_{11}[-1]) \dots (\alpha\tau + e_{NN}[-1]).$$

□

*Proof of Theorem 2.1.* Taking into account [2, Lemma 4.8.1], introduce the Lie superalgebra  $\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}_0 \oplus \widehat{\mathfrak{a}}_1$  such that  $\widehat{\mathfrak{a}}_0 = \widehat{\mathfrak{b}}$  is the Lie subalgebra of even elements, and  $\widehat{\mathfrak{a}}_1 = \mathfrak{m}[t, t^{-1}]$  is regarded as a supercommutative Lie superalgebra spanned by odd elements, while

$$[x, y] = \text{ad } x(y) \quad \text{for } x \in \widehat{\mathfrak{a}}_0 \quad \text{and} \quad y \in \widehat{\mathfrak{a}}_1.$$

We will use the notation  $\psi_{ji}[m] = e_{ji}t^{m-1}$  (with a standard shift of the power by 1) for the element  $e_{ji}t^{m-1} \in \mathfrak{m}[t, t^{-1}]$  when it is considered as an element of  $\widehat{\mathfrak{a}}_1$ .

Let  $V^k(\mathfrak{a})$  be the representation of  $\widehat{\mathfrak{a}}$  induced from the one-dimensional representation of  $(\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1}) \oplus \mathfrak{m}[t]$  on which  $\mathfrak{b}[t] \subset \widehat{\mathfrak{a}}_0$  and  $\mathfrak{m}[t] \subset \widehat{\mathfrak{a}}_1$  act trivially and  $\mathbf{1}$  acts as 1. Then  $V^k(\mathfrak{a})$  is naturally a vertex algebra which contains  $V^k(\mathfrak{b})$  as its vertex subalgebra. We will regard  $V^k(\mathfrak{a})$  as a (non-associative) algebra with respect to the  $(-1)$ -product

$$V^k(\mathfrak{a}) \otimes V^k(\mathfrak{a}) \rightarrow V^k(\mathfrak{a}), \quad a \otimes b \mapsto a_{(-1)}b, \quad (2.5)$$

where the Fourier coefficients  $a_{(n)}$  are defined in the usual way from the state-field correspondence map,

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \quad \text{for } a \in V^k(\mathfrak{a}).$$

The definition of the  $\mathcal{W}$ -algebra via the BRST cohomology [8, Ch. 15] can be stated in the form

$$\mathcal{W}^k(\mathfrak{g}) = \{v \in V^k(\mathfrak{b}) \mid Qv = 0\},$$

where  $Q : V^k(\mathfrak{a}) \rightarrow V^k(\mathfrak{a})$  is a derivation of the non-associative algebra  $V^k(\mathfrak{a})$  determined by the following properties. First,  $Q$  commutes with the translation operator  $D$  of the vertex algebra  $V^k(\mathfrak{a})$ . Moreover, we have the commutation relations

$$[Q, e_{ji}[-1]] = \sum_{a=i}^{j-1} e_{ai}[-1] \psi_{ja}[0] - \sum_{a=i+1}^j \psi_{ai}[0] e_{ja}[-1] + \alpha \psi_{ji}[-1] + \psi_{j+1i}[0] - \psi_{ji-1}[0],$$

which hold for all  $j \geq i$  (empty sums are understood as being equal to zero), and

$$[Q, \psi_{ji}[0]] = \frac{1}{2} \sum_{i < r < j} (\psi_{jr}[0] \psi_{ri}[0] - \psi_{ri}[0] \psi_{jr}[0]),$$

for  $j > i$ , where  $\psi_{kl}[r]$  is considered to be zero for out-of-range indices, and we omit the subscripts for the  $(-1)$ -products.

We will regard  $V^k(\mathfrak{a}) \otimes \mathbb{C}[\tau]$  as a non-associative algebra with the natural subalgebras  $V^k(\mathfrak{a})$  and  $\mathbb{C}[\tau]$  together with the relation  $[\tau, u] = D u$  for  $u \in V^k(\mathfrak{a})$ . In particular, the action of  $Q$  on the extended algebra commutes with the multiplication by  $\tau$ .

Now observe that the column-determinant  $\text{cdet } B$  which we defined over the associative algebra  $U(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$  coincides with the column-determinant  $\widetilde{\text{cdet}} B$  of the same matrix, but with the entries of  $B$  regarded as elements of the non-associative algebra  $V^k(\mathfrak{a}) \otimes \mathbb{C}[\tau]$ . Here we extend the definition of column-determinant to matrices  $A = [a_{ij}]$  with entries in a non-associative algebra by using right-normalized products,

$$\widetilde{\text{cdet}} A = \sum_{\sigma \in \mathfrak{S}_N} \text{sgn } \sigma \cdot a_{\sigma(1)1} (a_{\sigma(2)2} (a_{\sigma(3)3} \cdots (a_{\sigma(N-1)N-1} a_{\sigma(N)N}) \cdots)). \quad (2.6)$$

Furthermore, the products of the entries of  $B$  in the expression for  $\widetilde{\text{cdet}} B$  are actually associative due to the quasi-associativity property of the  $(-1)$ -product

$$(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c) + \sum_{j \geq 0} a_{(-j-2)}b_{(j)}c + \sum_{j \geq 0} b_{(-j-2)}a_{(j)}c \quad (2.7)$$

which holds for an arbitrary vertex algebra; see e.g. [9, Ch. 4]. Thus, the first part of the theorem will follow from the relation  $[Q, \widetilde{\text{cdet}} B] = 0$ . Expanding the column-determinant along the first column we get

$$\widetilde{\text{cdet}} B = \sum_{i=1}^N (\delta_{i1} \alpha \tau + e_{i1}[-1]) D^{(N-i)}, \quad (2.8)$$

where  $D^{(p)}$  denotes the column-determinant of the  $p \times p$  submatrix of  $B$  corresponding to the last  $p$  rows and columns (assuming  $D^{(0)} = 1$ ). Using the induction on  $N$ , we derive from the commutation relations satisfied by  $Q$  that

$$[Q, D^{(N-i)}] = - \sum_{j=i+1}^N \psi_{ji}[0] D^{(N-j)}.$$

Hence, by (2.8) we can write

$$\begin{aligned} [Q, \widetilde{\text{cdet}} B] &= \sum_{i=1}^N \left( \sum_{a=1}^{i-1} e_{a1}[-1] \psi_{ia}[0] - \sum_{a=2}^i \psi_{a1}[0] e_{ia}[-1] + \alpha \psi_{i1}[-1] + \psi_{i+1,1}[0] \right) D^{(N-i)} \\ &\quad - \sum_{i=1}^N (\delta_{i1} \alpha \tau + e_{i1}[-1]) \sum_{j=i+1}^N \psi_{ji}[0] D^{(N-j)}. \end{aligned}$$

Apply an appropriate super-version of the quasi-associativity (2.7) and change the order of summation in the second line to bring this expression to the form

$$- \sum_{i=2}^N \sum_{a=2}^i \psi_{a1}[0] e_{ia}[-1] D^{(N-i)} + \sum_{i=2}^N \psi_{i1}[0] D^{(N-i+1)} - \alpha \sum_{i=2}^N \psi_{i1}[0] \tau D^{(N-i)},$$

where we used the relation  $[\tau, \psi_{ji}[0]] = \psi_{ji}[-1]$ . Therefore, changing the order of summation in the first term, we obtain

$$[Q, \widetilde{\text{cdet}} B] = \sum_{i=2}^N \psi_{i1}[0] \left( D^{(N-i+1)} - \sum_{j=i}^N (\delta_{ji} \alpha \tau + e_{ji}[-1]) D^{(N-j)} \right) = 0,$$

by applying the first column expansion for  $D^{(N-i+1)}$ .

The second part of Theorem 2.1 follows by considering the grading of  $V^k(\mathfrak{b})$  induced by the principal grading of  $\mathfrak{b}$  defined by  $\deg e_{ij} = j - i$ . We have

$$W^{(r)} = \sum_{s=1}^{N-r+1} e_{r+s-1s}[-1] + \text{terms of higher degree.}$$

The elements  $\sum_{s=1}^{N-r+1} e_{r+s-1s}$  with  $r = 1, \dots, N$  form a basis of the centralizer  $\mathfrak{g}^f$ . Hence the argument is completed by applying [11, Theorem 4.1]; see also [1, Theorem 5.5.1].  $\square$

### 3 Rectangular $\mathcal{W}$ -algebras

Now we consider the general rectangular case where the Jordan type of a nilpotent element  $f \in \mathfrak{gl}_N$  is of the form  $(l^n)$  with  $nl = N$ . We identify  $\mathfrak{g} = \mathfrak{gl}_N$  with the tensor product of  $\mathfrak{gl}_l$  and  $\mathfrak{gl}_n$  via the isomorphism  $\mathfrak{gl}_l \otimes \mathfrak{gl}_n \rightarrow \mathfrak{g}$  defined by

$$e_{ij} \otimes e_{rs} \mapsto e_{(i-1)n+r, (j-1)n+s}, \quad (3.1)$$

where the  $e_{ij}$  denote the standard basis elements of the corresponding general linear Lie algebras. Set

$$f_l = \sum_{i=1}^{l-1} e_{i+1i} \in \mathfrak{gl}_l$$

and

$$f = f_l \otimes I_n = \sum_{i=1}^{l-1} \sum_{j=1}^n e_{in+j, (i-1)n+j} \in \mathfrak{g},$$

where  $I_n \in \mathfrak{gl}_n$  is the identity matrix. The matrix  $f$  is a nilpotent element of  $\mathfrak{g}$  of Jordan type  $(l^n)$ . Let

$$\mathfrak{gl}_l = \bigoplus_{p \in \mathbb{Z}} (\mathfrak{gl}_l)_p$$

be the standard principal grading of  $\mathfrak{gl}_l$ , obtained by defining the degree of  $e_{ij}$  to be equal to  $j - i$ . Set

$$\mathfrak{gl}_{l, \leq 0} = \bigoplus_{p \leq 0} (\mathfrak{gl}_l)_p \quad \text{and} \quad \mathfrak{gl}_{l, < 0} = \bigoplus_{p < 0} (\mathfrak{gl}_l)_p.$$

The isomorphism (3.1) then induces the  $\mathbb{Z}$ -grading on  $\mathfrak{g}$ ,

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p, \quad \mathfrak{g}_p = (\mathfrak{gl}_l)_p \otimes \mathfrak{gl}_n,$$

which is a *good grading* for  $f$  in the sense of [10]. We also set

$$\mathfrak{b} = \bigoplus_{p \leq 0} \mathfrak{g}_p = \mathfrak{gl}_{l, \leq 0} \otimes \mathfrak{gl}_n \quad \text{and} \quad \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p = \mathfrak{gl}_{l, < 0} \otimes \mathfrak{gl}_n. \quad (3.2)$$

For any  $k \in \mathbb{C}$ , consider the symmetric invariant bilinear form on  $\mathfrak{g}$  defined in (2.1) and for elements  $x, y \in \mathfrak{b}$  set

$$\kappa_{\mathfrak{b}}(x, y) = \kappa(x, y) + \frac{1}{2} \text{tr}_{\mathfrak{g}}(\text{ad } x \text{ ad } y) - \frac{1}{2} \text{tr}_{\mathfrak{g}_0} p_0(\text{ad } x \text{ ad } y),$$

where  $p_0$  denotes the restriction of the operator to  $\mathfrak{g}_0$ . Then  $\kappa_{\mathfrak{b}}$  defines a symmetric invariant bilinear form on  $\mathfrak{b}$ . For  $i \geq i'$  and  $j \geq j'$  we have

$$\begin{aligned} \kappa_{\mathfrak{b}}(e_{ii'} \otimes e_{pq}, e_{jj'} \otimes e_{rs}) \\ = \delta_{ii'} \delta_{jj'} \left( (k + nl) (\delta_{ij} \delta_{ps} \delta_{qr} - \frac{1}{nl} \delta_{pq} \delta_{rs}) - n \delta_{ij} (\delta_{ps} \delta_{qr} - \frac{1}{n} \delta_{pq} \delta_{rs}) \right). \end{aligned}$$

As with the principal case, we let  $\widehat{\mathfrak{b}} = \mathfrak{b}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$  be the Kac-Moody affinization of  $\mathfrak{b}$  with respect to the form  $\kappa_{\mathfrak{b}}$ , and let  $V^k(\mathfrak{b})$  be the universal affine vertex algebra associated with  $\mathfrak{b}$  and  $\kappa_{\mathfrak{b}}$  defined as in (2.2); see [9]. We have a vector space isomorphism  $V^k(\mathfrak{b}) \cong U(\mathfrak{b}[t^{-1}]t^{-1})$ . Due to [11] and [12] (see also [1]), the  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}, f)$  can be realized as a vertex subalgebra of  $V^k(\mathfrak{b})$ . We will give a description of the generators of  $\mathcal{W}^k(\mathfrak{g}, f)$  inside  $V^k(\mathfrak{b})$ . We will use the identification

$$\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n \cong \mathfrak{b}[t^{-1}]t^{-1},$$

defined by

$$e_{ji}[-m] \otimes e_{pq} \mapsto e_{(j-1)n+p, (i-1)n+q}[-m], \quad m \geq 1,$$

for  $1 \leq i \leq j \leq l$  and  $1 \leq p, q \leq n$ , where we write  $x[r] = x t^r$  for any  $r \in \mathbb{Z}$ .

By analogy with [5, Sec. 12], consider the tensor algebra  $T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1})$  of the vector space  $\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}$  and let  $M_n$  denote the matrix algebra with the basis formed by the matrix units  $e_{ij}$ ,  $1 \leq i, j \leq n$ . Define the algebra homomorphism

$$\mathcal{T} : T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \rightarrow M_n \otimes U(\mathfrak{b}[t^{-1}]t^{-1}), \quad x \mapsto \mathcal{T}(x) = \sum_{i,j=1}^n e_{ij} \otimes \mathcal{T}_{ij}(x)$$

by setting

$$\mathcal{T}_{ij}(x) = x \otimes e_{ji} \in \mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n = \mathfrak{b}[t^{-1}]t^{-1}$$

for  $x \in \mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}$ . By definition, for any  $x, y \in T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1})$  we have

$$\mathcal{T}_{ij}(xy) = \sum_{r=1}^n \mathcal{T}_{ir}(x) \mathcal{T}_{rj}(y) = \sum_{r=1}^n (x \otimes e_{ri})(y \otimes e_{jr}).$$

Let us equip the tensor product space  $T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$  with an associative algebra structure in such a way that the natural embeddings

$$T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \hookrightarrow T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \quad \text{and} \quad \mathbb{C}[\tau] \hookrightarrow T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$$

are algebra homomorphisms and the generator  $\tau$  satisfies the relations

$$[\tau, x[-m]] = mx[-m-1] \quad \text{for } x \in \mathfrak{gl}_{l, \leq 0} \quad \text{and } m \in \mathbb{Z}.$$

Furthermore, the tensor product space  $U(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$  will also be considered as an associative algebra in a similar way. We will extend  $\mathcal{T}$  to the algebra homomorphism

$$\mathcal{T} : T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \rightarrow M_n \otimes U(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$$

by setting  $\mathcal{T}_{ij}(uS) = \mathcal{T}_{ij}(u)S$  for  $u \in T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1})$  and any polynomial  $S \in \mathbb{C}[\tau]$ .

Set  $\alpha = k + n(l-1)$  and consider the matrix

$$B = \begin{bmatrix} \alpha\tau + e_{11}[-1] & -1 & 0 & \dots & 0 \\ e_{21}[-1] & \alpha\tau + e_{22}[-1] & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ e_{l-11}[-1] & e_{l-22}[-1] & \dots & \alpha\tau + e_{l-1l-1}[-1] & -1 \\ e_{l1}[-1] & e_{l2}[-1] & \dots & \dots & \alpha\tau + e_{ll}[-1] \end{bmatrix}$$

with entries in  $T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$ . Its column-determinant  $\text{cdet } B$  is defined by (2.3). So  $\text{cdet } B$  is an element of  $T(\mathfrak{gl}_{l, \leq 0}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau]$  and we can write

$$\mathcal{T}_{ij}(\text{cdet } B) = \sum_{r=0}^l W_{ij}^{(r)} (\alpha\tau)^{l-r}$$

for certain coefficients  $W_{ij}^{(r)}$  which are elements of  $U(\mathfrak{b}[t^{-1}]t^{-1})$ , and we can also regard them as elements of  $V^k(\mathfrak{b})$ . The following is our main result generalizing Theorem 2.1.

**Theorem 3.1.** *All coefficients  $W_{ij}^{(r)}$  belong to the  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}, f)$ . Moreover, the  $\mathcal{W}$ -algebra  $\mathcal{W}^k(\mathfrak{g}, f) \subset V^k(\mathfrak{b})$  is freely generated by the elements  $W_{ij}^{(r)}$  with  $1 \leq i, j \leq n$  and  $r = 1, 2, \dots, l$ .*



Set  $\mathfrak{l} = (\mathfrak{gl}_l)_0 \otimes \mathfrak{gl}_n \subset \mathfrak{gl}_N$ . As with the principal case, the projection  $\mathfrak{b} \rightarrow \mathfrak{l}$  induces the vertex algebra homomorphism  $V^k(\mathfrak{b}) \rightarrow V^k(\mathfrak{l})$ , which restricts to the *Miura transformation*

$$\nu : \mathcal{W}^k(\mathfrak{g}, f) \rightarrow V^k(\mathfrak{l}).$$

This is an injective vertex algebra homomorphism. The following formula for the images of the elements  $W_{ij}^{(r)}$  under the Miura transformation is an immediate consequence of Theorem 3.1.

**Corollary 3.2.** *We have*

$$\sum_{r=0}^l \nu(W_{ij}^{(r)})(\alpha\tau)^{l-r} = \mathcal{T}_{ij} \left( (\alpha\tau + e_{11}[-1]) \dots (\alpha\tau + e_{ll}[-1]) \right).$$

The principal  $\mathcal{W}$ -algebra corresponds to the case  $n = 1$  (and  $N = l$ ) so that Corollary 3.2 generalizes the Fateev–Lukyanov formula; see Corollary 2.2. For a concrete example in the non-principal case see Example 3.3 below.

*Proof of Theorem 3.1.* Recall the notation (3.2) and let  $\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}_0 \oplus \widehat{\mathfrak{a}}_1$  be the Lie superalgebra such that elements of  $\widehat{\mathfrak{a}}_0 = \widehat{\mathfrak{b}}$  are even, and elements of  $\widehat{\mathfrak{a}}_1 = \mathfrak{m}[t, t^{-1}]$  are odd, where  $\mathfrak{m}[t, t^{-1}]$  is regarded as the supercommutative Lie superalgebra, while

$$[x, y] = \text{ad } x(y) \quad \text{for } x \in \widehat{\mathfrak{a}}_0 \quad \text{and} \quad y \in \widehat{\mathfrak{a}}_1.$$

We will write  $\psi_{ji}[m] \otimes e_{pq}$  for the element

$$e_{ji}t^{m-1} \otimes e_{pq} \in \mathfrak{gl}_{l, < 0}[t, t^{-1}] \otimes \mathfrak{gl}_n = \mathfrak{m}[t, t^{-1}]$$

when it is considered as an element of  $\widehat{\mathfrak{a}}_1$ .

Let  $V^k(\mathfrak{a})$  be the representation of  $\widehat{\mathfrak{a}}$  induced from the one-dimensional representation of  $(\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1}) \oplus \mathfrak{m}[t]$  on which  $\mathfrak{b}[t] \subset \widehat{\mathfrak{a}}_0$  and  $\mathfrak{m}[t] \subset \widehat{\mathfrak{a}}_1$  act trivially and  $\mathbf{1}$  acts as 1. Then  $V^k(\mathfrak{a})$  is naturally a vertex algebra which contains  $V^k(\mathfrak{b})$  as its vertex subalgebra. Endow  $V^k(\mathfrak{a})$  with the  $(-1)$ -product defined as in (2.5) so that it gets a structure of a non-associative algebra. By [11] and [12] the  $\mathcal{W}$ -algebra is given by

$$\mathcal{W}^k(\mathfrak{g}, f) = \{v \in V^k(\mathfrak{b}) \mid Qv = 0\},$$

where  $Q : V^k(\mathfrak{a}) \rightarrow V^k(\mathfrak{a})$  is the derivation of the non-associative algebra  $V^k(\mathfrak{a})$  defined by the following properties. The map  $Q$  commutes with the translation operator  $D$  of the vertex algebra  $V^k(\mathfrak{a})$  and we have the commutation relations

$$\begin{aligned} [Q, e_{ji}[-1] \otimes e_{pq}] &= \sum_{a=i}^{j-1} \sum_{r=1}^n (e_{ai}[-1] \otimes e_{rq}) (\psi_{ja}[0] \otimes e_{pr}) \\ &\quad - \sum_{a=i+1}^j \sum_{r=1}^n (\psi_{ai}[0] \otimes e_{rq}) (e_{ja}[-1] \otimes e_{pr}) \\ &\quad + \alpha \psi_{ji}[-1] \otimes e_{pq} + \psi_{j+1i}[0] \otimes e_{pq} - \psi_{ji-1}[0] \otimes e_{pq} \end{aligned}$$

and

$$\begin{aligned} [Q, \psi_{ji}[0] \otimes e_{pq}] &= \frac{1}{2} \sum_{i < r < j, 1 \leq s \leq n} (\psi_{jr}[0] \otimes e_{sq}) (\psi_{ri}[0] \otimes e_{ps}) \\ &\quad - \frac{1}{2} \sum_{i < r < j, 1 \leq s \leq n} (\psi_{ri}[0] \otimes e_{sq}) (\psi_{jr}[0] \otimes e_{ps}) \end{aligned}$$

with  $\alpha = k + n(l - 1)$ . Here we assumed that  $\psi_{kl}[r] = 0$  for out-of-range subscripts and used the fact that

$$\mathrm{tr}_{\mathfrak{m}} p_+ (\mathrm{ad}(e_{ji} \otimes e_{pq}) \mathrm{ad}(e_{ij} \otimes e_{pq})) = n(l + i - j - 1)$$

for  $1 \leq i < j \leq l$  and  $1 \leq p, q \leq n$ , where  $p_+$  denotes the restriction of the operator to  $\mathfrak{m}$ .

Our goal now is to reduce the calculations to the principal nilpotent case. To this end, when  $n = 1$  we will write  $\bar{\mathfrak{a}}$  and  $\bar{\mathfrak{b}}$  respectively, instead of  $\mathfrak{a}$  and  $\mathfrak{b}$ , and replace  $k$  with  $\bar{k} = k + (n - 1)(l - 1)$  in (2.1). Consequently,  $V^{\bar{k}}(\bar{\mathfrak{a}})$  will denote the vertex algebra  $V^k(\mathfrak{a})$  with  $n = 1$  (and  $k$  replaced by  $\bar{k}$ ). We let  $\bar{Q}$  denote the operator  $Q$  for  $V^{\bar{k}}(\bar{\mathfrak{a}})$ . The commutation relations for  $\bar{Q}$  are given in the proof of Theorem 2.1.

We will regard  $V^k(\mathfrak{a}) \otimes \mathbb{C}[\tau]$  as a non-associative algebra with the natural subalgebras  $V^k(\mathfrak{a})$  and  $\mathbb{C}[\tau]$  together with the relation  $[\tau, u] = Du$  for  $u \in V^k(\mathfrak{a})$ . Similarly, the tensor product  $V^{\bar{k}}(\bar{\mathfrak{a}}) \otimes \mathbb{C}[\tau]$  will be regarded as a non-associative algebra with the relation  $[\tau, u] = \bar{D}u$  for  $u \in V^{\bar{k}}(\bar{\mathfrak{a}})$ , where  $\bar{D}$  denotes the translation operator of the vertex algebra  $V^{\bar{k}}(\bar{\mathfrak{a}})$ . Define the non-associative algebra homomorphism

$$\tilde{\mathcal{T}} : V^{\bar{k}}(\bar{\mathfrak{a}}) \otimes \mathbb{C}[\tau] \rightarrow M_n \otimes V^k(\mathfrak{a}) \otimes \mathbb{C}[\tau], \quad x \mapsto \tilde{\mathcal{T}}(x) = \sum_{p,q=1}^n e_{pq} \otimes \tilde{\mathcal{T}}_{pq}(x)$$

by

$$\tilde{\mathcal{T}}_{pq}(e_{ji}[m]) = e_{ji}[m] \otimes e_{qp}, \quad \tilde{\mathcal{T}}_{pq}(\psi_{ji}[m]) = \psi_{ji}[m] \otimes e_{qp} \quad \text{and} \quad \tilde{\mathcal{T}}_{pq}(\tau) = \tau.$$

Note the relation  $\tilde{\mathcal{T}}(\widetilde{\mathrm{cdet}} B) = \mathcal{T}(\mathrm{cdet} B)$ , where  $\widetilde{\mathrm{cdet}} B$  is defined by (2.6) and regarded as an element of  $V^{\bar{k}}(\bar{\mathfrak{a}}) \otimes \mathbb{C}[\tau]$ . The first part of Theorem 3.1 will now be implied by the property

$$[Q, \tilde{\mathcal{T}}_{pq}(a)] = \tilde{\mathcal{T}}_{pq}([\bar{Q}, a])$$

which holds for any  $a \in V^{\bar{k}}(\bar{\mathfrak{a}}) \otimes \mathbb{C}[\tau]$  and  $1 \leq p, q \leq n$ , and which follows from the definitions of the operators  $Q$  and  $\bar{Q}$ . It remains to note that the relation  $[\bar{Q}, \widetilde{\mathrm{cdet}} B] = 0$  was already verified in the proof of Theorem 2.1.

To prove the second part of Theorem 3.1, consider the grading of  $V^k(\mathfrak{b})$  induced by the grading of  $\mathfrak{b}$ . One has

$$W_{ij}^{(r)} = \mathcal{T}_{ij} \left( \sum_{s=1}^{l-r+1} e_{r+s-1s}[-1] \right) + \text{terms of higher degree.}$$

The elements  $\sum_{s=1}^{l-r+1} e_{r+s-1s}$  with  $r = 1, \dots, l$  form a basis of  $\mathfrak{gl}_l^{f_i}$  and the elements

$$\sum_{s=1}^{l-r+1} e_{r+s-1s} \otimes e_{ji}, \quad r = 1, \dots, l \quad \text{and} \quad i, j = 1, \dots, n,$$

form a basis of  $\mathfrak{g}^f$ . It remains to apply [11, Theorem 4.1]; see also [1, Theorem 5.5.1].  $\square$

*Example 3.3.* Take  $n = l = 2$  so that  $N = 4$ . We have

$$\text{cdet } B = (\alpha\tau)^2 + (e_{11}[-1] + e_{22}[-1])(\alpha\tau) + e_{11}[-1]e_{22}[-1] + e_{21}[-1] + \alpha e_{22}[-2]$$

with  $\alpha = k + 2$ . Hence

$$\begin{aligned} W_{11}^{(1)} &= e_{11}[-1] + e_{33}[-1], & W_{22}^{(1)} &= e_{22}[-1] + e_{44}[-1], \\ W_{21}^{(1)} &= e_{12}[-1] + e_{34}[-1], & W_{12}^{(1)} &= e_{21}[-1] + e_{43}[-1], \\ W_{11}^{(2)} &= e_{11}[-1]e_{33}[-1] + e_{21}[-1]e_{34}[-1] + e_{31}[-1] + \alpha e_{33}[-2], \\ W_{22}^{(2)} &= e_{12}[-1]e_{43}[-1] + e_{22}[-1]e_{44}[-1] + e_{42}[-1] + \alpha e_{44}[-2], \\ W_{21}^{(2)} &= e_{12}[-1]e_{33}[-1] + e_{22}[-1]e_{34}[-1] + e_{32}[-1] + \alpha e_{34}[-2], \\ W_{12}^{(2)} &= e_{11}[-1]e_{43}[-1] + e_{21}[-1]e_{44}[-1] + e_{41}[-1] + \alpha e_{43}[-2]. \end{aligned}$$

For the images under the Miura transformation we have

$$\begin{aligned} \nu(W_{11}^{(1)}) &= e_{11}[-1] + e_{33}[-1], & \nu(W_{22}^{(1)}) &= e_{22}[-1] + e_{44}[-1], \\ \nu(W_{21}^{(1)}) &= e_{12}[-1] + e_{34}[-1], & \nu(W_{12}^{(1)}) &= e_{21}[-1] + e_{43}[-1], \\ \nu(W_{11}^{(2)}) &= e_{11}[-1]e_{33}[-1] + e_{21}[-1]e_{34}[-1] + \alpha e_{33}[-2], \\ \nu(W_{22}^{(2)}) &= e_{12}[-1]e_{43}[-1] + e_{22}[-1]e_{44}[-1] + \alpha e_{44}[-2], \\ \nu(W_{21}^{(2)}) &= e_{12}[-1]e_{33}[-1] + e_{22}[-1]e_{34}[-1] + \alpha e_{34}[-2], \\ \nu(W_{12}^{(2)}) &= e_{11}[-1]e_{43}[-1] + e_{21}[-1]e_{44}[-1] + \alpha e_{43}[-2]. \end{aligned}$$

The values of the form  $\kappa_b(x, y)$  are given in the following table, where the columns and rows correspond to the  $x$  and  $y$  variables, respectively:

	$e_{11}$	$e_{22}$	$e_{33}$	$e_{44}$	$e_{12}$	$e_{21}$	$e_{34}$	$e_{43}$
$e_{11}$	$\frac{3k+8}{4}$	$-\frac{k}{4}$	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	0	0	0	0
$e_{22}$	$-\frac{k}{4}$	$\frac{3k+8}{4}$	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	0	0	0	0
$e_{33}$	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	$\frac{3k+8}{4}$	$-\frac{k}{4}$	0	0	0	0
$e_{44}$	$-\frac{k+4}{4}$	$-\frac{k+4}{4}$	$-\frac{k}{4}$	$\frac{3k+8}{4}$	0	0	0	0
$e_{12}$	0	0	0	0	0	$k+2$	0	0
$e_{21}$	0	0	0	0	$k+2$	0	0	0
$e_{34}$	0	0	0	0	0	0	0	$k+2$
$e_{43}$	0	0	0	0	0	0	$k+2$	0

These values can be used to calculate the operator product expansion formulas for the generators of  $\mathcal{W}^k(\mathfrak{g}, f)$ . In particular, set

$$L = \frac{1}{2(k+4)} \left( -2(W_{11}^{(2)} + W_{22}^{(2)}) + W_{12}^{(1)}W_{21}^{(1)} + \frac{3}{4}(W_{11}^{(1)}W_{11}^{(1)} + W_{22}^{(1)}W_{22}^{(1)}) \right. \\ \left. - \frac{1}{2}W_{11}^{(1)}W_{22}^{(1)} - (k+2)(W_{11}^{(1)} + W_{22}^{(1)})' - (W_{11}^{(1)} - W_{22}^{(1)})' \right),$$

where the primes indicate the action of  $\text{ad } \tau$  taking  $e_{ij}[-1]$  to  $e_{ij}[-2]$ . Then  $L$  is the conformal vector of  $\mathcal{W}^k(\mathfrak{g}, f)$ :

$$L(z)L(w) \sim -\frac{12k^2 + 41k + 32}{2(k+4)^2(z-w)^4} + \frac{2}{(z-w)^2}L(w) + \frac{1}{z-w}\partial L(w).$$

## References

- [1] T. Arakawa, *Representation theory of superconformal algebras and the Kac–Roan–Wakimoto conjecture*, Duke Math. J. **130** (2005), 435–478.
- [2] T. Arakawa, *Representation theory of  $W$ -algebras*, Invent. Math. **169** (2007), 219–320.
- [3] T. Arakawa, *Rationality of  $W$ -algebras: principal nilpotent cases*, Ann. of Math. (2) **182** (2015), 565–604.
- [4] C. Briot and E. Ragoucy, *RTT presentation of finite  $\mathcal{W}$ -algebras*, J. Phys. A **34** (2001), 7287–7310.
- [5] J. Brundan and A. Kleshchev, *Shifted Yangians and finite  $W$ -algebras*, Adv. Math. **200** (2006), 136–195.
- [6] V. A. Fateev and S. L. Lukyanov, *The models of two-dimensional conformal quantum field theory with  $Z_n$  symmetry*, Internat. J. Modern Phys. A **3** (1988), 507–520.
- [7] B. Feigin and E. Frenkel, *Quantization of the Drinfeld–Sokolov reduction*, Phys. Lett. B **246** (1990), 75–81.
- [8] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Second edition. Mathematical Surveys and Monographs, 88. AMS, Providence, RI, 2004.
- [9] V. Kac, *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, Providence, RI, 1997.
- [10] V. Kac, Shi-Shyr Roan and M. Wakimoto, *Quantum reduction for affine superalgebras*, Comm. Math. Phys. **241** (2003), 307–342.
- [11] V. Kac and M. Wakimoto, *Quantum reduction and representation theory of superconformal algebras*, Adv. Math. **185** (2004), 400–458.

- [12] V. Kac and M. Wakimoto, *Corrigendum to: “Quantum reduction and representation theory of superconformal algebras”* [*Adv. Math.* **185** (2004), 400–458], *Adv. Math.* **193** (2005), 453–455.
- [13] E. Ragoucy and P. Sorba, *Yangian realisations from finite  $\mathcal{W}$ -algebras*, *Comm. Math. Phys.* **203** (1999), 551–572.